$\mathcal{D}$-modules - Tutorial 1

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Example 1. *Let $\mathcal{D}_n$ be the Weyl algebra then for every $s \in \mathbb{Z}_{>0}$ we can define a filtration $F_s^k\mathcal{D}_n := \{\Sigma_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta : (\alpha,\beta) \in \mathbb{N}^{\times 2n}\}, |\alpha| + s|\beta| \leq k\}$ (where $|\alpha| = \Sigma_j \alpha_j$). For this filtration $gr_{F_s}\mathcal{D}_n = k[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$ with $deg(x_j) = 1$ and $deg(\xi_j) = s$ (exercise).*

Notice that the case $s = 1$ in the example above corresponds to the Bernstein filtration. Also note how when $s \geq 2$ this filtration is not good for our definitions (essentially because it isn't generated in degree 1). Although the associated graded is still commutative noetherian. This filtration is useful because it interpolates between the arithmetic and the geometric filtration. As s tends to ∞ the symbol map looks more like the geometric symbol and less like the arithmetic one.

lemma 1. *Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence of integers. The following are equivalent:*

1. *f is eventually polynomial of degree d*
2. *$\Delta[f](n-1) := f(n) - f(n-1)$ is eventually polynomial of degree $d-1$*
3. *$f(n) \sim \Sigma_{j=0}^d a_j \binom{n}{j}$ for some $a_j \in \mathbb{Z}$*
4. *The "poincare series" satisfies $P(z) := \Sigma_n f(n) z^n = R(z) + \frac{1}{1-z} Q(\frac{z}{1-z})$ where $R \in \mathbb{Z}[z]$ and $Q \in \mathbb{Z}[z]$ with $deg(Q) = d$.*

Proof. The implication (1) $\iff$ (2) is a direct computation. We show that together they imply 3. Observe that $f(n) = f(n-1) + \Delta f(n-1) = (I+\Delta)[f](n-1) = (I+\Delta)^k[f](n-k) = \Sigma_{j=0}^k \binom{k}{j} \Delta^j[f](n-k)$. By assumption $\Delta^j[f](m) = 0$ for $j > d$ and large m. Let $n > n_0 >> 0$ and observe that

$$f(n) = \Sigma_{j=0}^{n-n_0} \binom{n-n_0}{j} \Delta^j[f](n_0) = \Sigma_{j=0}^{d} \binom{n-n_0}{j} \Delta^j[f](n_0)$$

In other words we have shown that $f(n+n_0) = \Sigma_{j=0}^d \binom{n}{j} \Delta^j[f](n_0)$. We are almost done, we only need to get rid of the pesky n_0. To do so we proceed by induction on d.
The base case ($d = 0$) is trivial so let us assume the statement holds for all polynomials of degree less than d. Observe now that $g(n) = f(n) - f(n+n_0)$ must be eventually a polynomial of degree smaller then f (as translation doesn't change the highest order term). Therefore combining the induction hypothesis and the statement we showed earlier we are done.
Now we show that (3) $\implies$ (4). We can replace the $\sim$ symbol by an $=$ sign if we add to f some function g which is eventually 0. The poincare series for this g will be a polynomial. This is where the $R(z)$ comes from. We are left to prove the statement in the case where $R = 0$ and in (3) we have an equality. By linearity its enough to show that $\Sigma_n \binom{n}{j} z^n = \frac{t^j}{(1-z)^{j+1}}$ which is a direct computation. Finally (4) $\implies$ (1) follows from the same argument traced backwards.

□

Recall. *If (M, F) be a good filtered A-module. The function $h_M(n) = dim(F^n M)$ is eventually polynomial. Let $e_d\binom{n}{d}$ be the "leading term" of h_M then we denote $d(M) = d$ and $e(M) = e$.*

Note that we can interpret these also as $d(M) = ord_{z=1}(P_M(z)) - 1$ (order means order of the pole - not zero) and $e(M) = Res_{z=1}[(1-z)^{d(M)} P_M(z)]$. The point of this is that we know $res(P+Q) = res(P) + res(Q)$ and $ord(P+Q) \leq max\{ord(P), ord(Q)\}$. But in fact in our case (where we start with $f : \mathbb{N} \to \mathbb{N}$) its always true that the coefficient of the lowest power of $(1-z)$ in P is positive so their can be no cancellations. Thus for $P(z), Q(z)$ which come from poincare series of good filtered modules we know that $ord(P+Q) = max\{ord(P), ord(Q)\}$.

Exercise 1. *Consider an exact sequence of good filtered A-modules (with strict maps)*

$$0 \to L \to M \to N \to 0$$

Then $h_M(n) = h_L(n) + h_N(n)$.

Proof. Follows from additivity of dimension in short exact sequences □

Corollary 1. *In the situation above $d(M) = \max\{d(L), d(N)\}$ and*

$$e(M) = \begin{cases} e(N) & d(L) < d(N) \\ e(N) + e(L) & d(L) = d(N) \\ e(L) & d(N) < d(L) \end{cases}$$

Proof. Follows from the description of $d(M)$ and $e(M)$ as order of pole and residue respectively. □

We give some simple examples of $\mathcal{D}$-modules in dimension 1 so we fix $\mathcal{D} = \mathcal{D}_1$

Example 2. *Polynomials $k[x] = \mathcal{D}/\mathcal{D}\partial$ form an irreducible module, since we can get to the generator 1 from any other non-zero polynomial.*

Example 3. *Let $g \in k[x]$ be any polynomial then the ring $k[x]_{(g)}$ (in which g is inverted) is also a $\mathcal{D}$-module (via the usual action of differential operators on functions).*

Example 4. *The module $\delta := \mathcal{D}/\mathcal{D}x$ looks like $k[\partial]$ as a vector space. This module is also irreducible (essentially by the same argument as before). It corresponds in some sense (which we will make precise later in the course perhaps) to the delta function. In this module ∂^n can be interpreted as $\delta^{(n)}(x)$ (the nth "weak" derivative of the delta function).*

Example 5. *Let $M_\alpha = D/D(x\partial - \alpha)$. Here are some facts about this module*

1. *$M_\alpha \cong M_\beta \iff \alpha - \beta \in \mathbb{Z}$*
2. *M_α is reducible iff $\alpha \in \mathbb{Z}$ (so when $k = \mathbb{C}$ we have $\mathbb{C}/\mathbb{Z} = \mathbb{C}^*$ worth of distinct such modules)*
3. *When $\alpha \in \mathbb{Z}$ we have a short exact sequence*

$$0 \to k[x] \to M_\alpha \to \delta \to 0$$

The following diagram of all weight spaces of $x\partial$ acting on M_α may be useful in convincing oneself of the above statements.

$$\cdots \underset{\partial}{\overset{x}{\rightleftarrows}} V_{\alpha-1} \underset{\partial}{\overset{x}{\rightleftarrows}} V_\alpha \underset{\partial}{\overset{x}{\rightleftarrows}} V_{\alpha+1} \underset{\partial}{\overset{x}{\rightleftarrows}} \cdots$$